UNIFORM PERFECTNESS OF THE LIMIT SETS AND TRANSLATION LENGTHS OF KLEINIAN GROUPS(Analysis of Discrete Groups II)

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UNIFORM PERFECTNESS OF THE LIMIT SETS AND TRANSLATION LENGTHS OF KLEINIAN GROUPS

須川 敏幸 (京都大学・理学部)
TOSHIYUKI SUGAWA (KYOTO UNIVERSITY)

1. Introduction

In this note, we shall consider the uniform perfectness of the limit sets of Kleinian groups. As soon as the proof of uniform perfectness of the limit sets of Schottky groups appeared in [1], this result was generalized to the case of finitely generated non-elementary Kleinian groups by Maskit, Marden, etc. (cf. [8] and [9]). Afterward, Canary remarked in [3] that the same result holds for analytically finite Kleinian groups.

Recently, Järvi and Vuorinen [4] proved the uniform perfectness for finitely generated Kleinian groups in the higher dimensional case. It is noteworthy that their proof does not rely on Ahlfors' Finiteness Theorem.

In this note, we will present a more general condition for the uniform perfectness of the limit set (Theorem 3.3 or Corollary 3.4). Our method also provides a bound for uniform perfectness by some geometric quantity. In practice, it is important to know an explicit bound because the uniform perfectness connects with various quantity relating to the geometry of the quotient surface (cf. Sugawa [10]). Indeed, we shall give some applications in Sections 5 and 6. In Section 5, we state results relating to the regularity in the sense of Dirichlet and the Hausdorff dimention of the limit sets. Section 6 is devoted to an estimate of translation length by the trace (or the multiplier) of a loxodromic element of a Kleinian group. We shall conclude this note by giving an example of those Kleinian groups whose limit sets are not uniformly perfect.

2. Riemann orbifolds

A 1-dimensional complex orbifold is a Hausdorff topological space locally modeled on the quotient of an open subset of the complex plane $\mathbb{C}$ by an action of finite group consisting of biholomorphic maps. By a Riemann orbifold (or, an orbifold, for short), we will mean a connected 1-dimensional complex orbifold. For a precise definition and fundamental properties of orbifolds, we refer the reader to McMullen's

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book [6]. Let $X = (R, \nu)$ be a hyperbolic orbifold, in other words, X is a pair of a Riemann surface R and a multiplicity map $\nu : R \to \mathbb{N}$ such that the set of singularities (or, branch points) $b(X) = \{x \in R; \nu(x) > 1\}$ is discrete, and R has a branched holomorphic normal covering map $p : \mathbb{H} \to R$ from the upper half plane $\mathbb{H}$ onto R such that $\mathrm{ord}_\zeta p = \nu(p(\zeta))$ for any $\zeta \in \mathbb{H}$. We set $\overset{\circ}{X} = R \setminus b(R)$. Then $\overset{\circ}{X}$ can be regarded as a usual Riemann surface. If N is the constant 1, then X can be naturally indentified with the underlying Riemann surface R. The covering transformation group $\Gamma = \Gamma_X = \{\gamma \in Aut(\mathbb{H}); p \circ \gamma = p\}$ is called a Fuchsian model of X. Through this universal covering map, the Riemann orbifold X inherits the hyperbolic metric ρ_X from $\mathbb{H}$, i.e., $\rho_{\mathbb{H}} = \frac{|dz|}{2\mathrm{Im}z} = p^*(\rho_X)$. We denote by d_X the distance function on X naturally defined by the hyperbolic metric.

Let $\mathcal{C}_X$ denote the set of free homotopy classes $[\alpha]$ of non-trivial closed curves α in X, where a curve α is said to be non-trivial if it is covered by an element $\gamma \in \Gamma$ of infinite order, more precisely, there exists a lift $\tilde{\alpha}$ via p such that the initial point is translated to the terminal one by γ, and said to be freely homotopic to another α' if both are covered by the same element of Γ. (The free homotopy class precisely corresponds to the conjugacy class of the element of Γ which covers the curve and has infinite order.) And let $\mathcal{C}_X^*$ be the set of free homotopy classes which are covered by hyperbolic elements of Γ.

We write as $\ell_X(\alpha) = \int_\alpha \rho_X$ and $\ell_X[\alpha] = \inf_{\alpha' \in [\alpha]} \ell_X(\alpha')$. By definition, if $\gamma \in \Gamma$ covers α, then

$$\ell_X[\alpha] = l_\gamma = \cosh^{-1}\left(\frac{\mathrm{tr}\gamma}{2}\right), \tag{2.1}$$

where l_γ denotes the translation length of γ and $\mathrm{tr}\gamma$ is the trace of a representative of γ in $\mathrm{SL}(2, \mathbb{R})$.

Now we set

$$L(X) = \inf_{[\alpha] \in \mathcal{C}_X} \ell_X[\alpha] \quad \text{and} \quad L^*(X) = \inf_{[\alpha] \in \mathcal{C}_X^*} \ell_X[\alpha].$$

We call X is *modulated* if $L(X) > 0$. We note here that if X is a Riemann surface R, the constant $L(R)$ is nothing but $2I_R$, where I_R is the injectivity radius of R (see [10]).

Let $A_2(X)$ and $B_2(X)$ be the complex Banach spaces consisting of holomorphic quadratic differentials φ on X with norms

$$\|\varphi\|_1 = \iint_X |\varphi| = \iint_X |\varphi(z)| dxdy,$$
$$\|\varphi\|_\infty = \sup \rho_X^{-2} |\varphi| = \sup \rho_X^{-2}(z) |\varphi(z)|,$$

respectively. The spaces $A_2(X)$ and $B_2(X)$ are canonically isomorphic to the spaces $A_2(\mathbb{H}, \Gamma)$ and $B_2(\mathbb{H}, \Gamma)$ of integrable and bounded holomorphic automorphic forms of weight -4 on $\mathbb{H}$ for Γ, respectively.

And we set $\kappa(X) = \sup\{\|\varphi\|_\infty; \varphi \in A_2(X) \text{ with } \|\varphi\| \leq 1\}$. For these spaces, the inclusion problem was first settled by Niebur and Sheingorn [7]. The following strong form is due to Matsuzaki [5].

Theorem 2.1. *The space $A_2(X)$ is (continuously) included by $B_2(X)$ if and only if $L^*(X) > 0$. Furthermore, there exist universal constants r_0 and r_1 such that*

$$\frac{1}{2\pi L^*(\mathring{X})} \leq \kappa(X) \leq \max\{\frac{r_0}{L^*(X)}, r_1\}.$$

Note that the inclusion map $\mathring{X} \hookrightarrow X$ induces an isometric isomorphism $A_2(X) \to A_2(\mathring{X})$ and a bounded linear operator $B_2(X) \to B_2(\mathring{X})$ with $\|\varphi\|_{B_2(\mathring{X})} \leq \|\varphi\|_{B_2(X)}$ by the monotonicity of hyperbolic metrics: $\rho_X \leq \rho_{\mathring{X}}$. Therefore, we have $\kappa(\mathring{X}) \leq \kappa(X)$. In fact, the following holds.

Proposition 2.2. $\kappa(\mathring{X}) \leq \kappa(X) \leq 3\kappa(\mathring{X})$.

Proof. It suffices to show that $\kappa(X) \leq 3\kappa(\mathring{X})$. Let X^* be the mirror image $\mathbb{H}^*/\widehat{\Gamma}$ of X, where $\mathbb{H}^*$ denotes the lower half plane. By the Bers-Greenberg isomorphism theorem [2], the inclusion map $\bar{\jmath} : \mathring{X}^* \hookrightarrow X^*$ induces the natural biholomorphism $\bar{\jmath}^\# : T(X^*) \to T(\mathring{X}^*)$ between Teichmüller spaces of X^* and $\mathring{X}^*$. Through the Bers embedding, we regard the Teichmüller spaces $T(X^*)$ and $T(\mathring{X}^*)$ are embedded in $B_2(X)$ and $B_2(\mathring{X})$, respectively. Here, we note that the Fréchet derivative $d_0\bar{\jmath}^\# : B_2(X) \to B_2(\mathring{X})$ of $\bar{\jmath}^\#$ at 0 is nothing but the pull-back j^* by the inclusion map $j : \mathring{X} \hookrightarrow X$ as quadratic differentials. The Ahlfors-Weill theorem and the Nehari-Kraus theorem imply that $B_2(\mathring{X})_2 \subset T(\mathring{X}^*)$ and $T(X^*) \subset B_2(X)_6$, thus $(\bar{\jmath}^\#)^{-1}(B_2(\mathring{X})_2) \subset B_2(X)_6$, where $B_2(*)_r$ denotes the open ball of the Banach space $B_2(*)$ centered at 0 and with radius r as usual. Since $(\bar{\jmath}^\#)^{-1}$ maps 0 to 0, Schwarz's lemma yields that $\|d_0(\bar{\jmath}^\#)^{-1}\| = \|(d_0\bar{\jmath}^\#)^{-1}\| = \|(j^*)^{-1}\| \leq 3$. This is equivalent to the desired assertion. $\square$

In this article, we will say that X is *of Lehner type* if $L^*(X) > 0$. Generally, for (possibly disconnected) 1-dimensional complex orbifold X, we define $L(X) = \inf L(X_0)$ and $L^*(X) = \inf L^*(X_0)$, where the infima are taken over all connected components X_0 of X, and we say that X is modulated or of Lehner type if $L(X) > 0$ or $L^*(X) > 0$, respectively.

A closed set C in the Riemann sphere $\widehat{\mathbb{C}}$ with $\#C \geq 3$ is called *uniformly perfect* if the complement $R = \widehat{\mathbb{C}} \setminus C$ is modulated.

3. Kleinian groups and main results

Let G be a Kleinian group acting on the Riemann sphere $\widehat{\mathbb{C}}$, i.e., G is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ whose region of discontinuity $\Omega(G)$ is not empty. We denote by $\Lambda(G)$ the limit set of G, i.e., $\Lambda(G) = \widehat{\mathbb{C}} \backslash \Omega(G)$. In the following, we always assume that G is non-elementary, in ohter words, $\#\Lambda(G) \geq 3$. Then $\Lambda(G)$ is known to be a perfect set. The quotient space $X(G) = \Omega(G)/G$ has a natural hyperbolic 1-dimensional complex orbifold structure with which the canonical projection $\pi : \Omega(G) \to X(G)$ is a covering map.

Let X be a connected component of $X(G)$ and Ω be a connected component of $\pi^{-1}(X)$. If $q : \mathbb{H} \to \Omega$ is a holomorphic universal covering map, then clearly $p = \pi \circ q$ is a holomorphic universal covering map for X. Let $H = H_\Omega$ be the component subgroup of G corresponding to Ω, i.e., $H = \mathrm{Stab}_G(\Omega) = \{g \in G; g(\Omega) = \Omega\}$. And denote by Γ and $\widetilde{\Gamma}$ the covering transformation groups of q and p, respectively. Then, we have a natural exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \widetilde{\Gamma} \xrightarrow{\ \chi\ } H \longrightarrow 1$$

of group homomorphisms, here χ can be described as $q \circ \gamma = \chi(\gamma) \circ q$ for all $\gamma \in \widetilde{\Gamma}$.

For each $h \in H$, we denote by l_h the translation length of h in Ω :

$$l_h := \inf_{z \in \Omega} d_\Omega(z, h(z)). \tag{3.1}$$

Let

$$\begin{aligned} H_e &= \{h \in H; h \text{ has a fixed point in } \Omega\} \\ &= \{h \in H; h \text{ is the identity or elliptic and } l_h = 0\} \text{ and} \\ H_p &= \{h \in H; h \text{ corresponds to a puncture of } X = \Omega/H\} \\ &= \{h \in H; h \text{ is parabolic and } l_h = 0\}. \end{aligned}$$

We remark that there may exist an elliptic or parabolic element h of H with $l_h > 0$. Such a parabolic element is sometimes called an accidental parabolic transformation. Now we define λ_H, λ_H^* as follows:

$$\lambda_H = \inf_{h \in H \backslash H_e} l_h, \quad \text{and} \quad \lambda_H^* = \inf_{h \in H \backslash (H_e \cup H_p)} l_h.$$

Noting that an elliptic or parabolic element in a Fuchsian group always represents a branch point or a puncture of the quotient surface, similarly we set

$$\lambda_{\widetilde{\Gamma}} = \inf_{\widetilde{\Gamma} \ni h:\ \text{of infinite order}} l_h \quad \text{and} \quad \lambda_{\widetilde{\Gamma}}^* = \inf_{\widetilde{\Gamma} \ni h:\ \text{hyperbolic}} l_h.$$

We can define $\lambda_\Gamma, \lambda_\Gamma^*$ in the same manner, however it holds that $\lambda_\Gamma = \lambda_\Gamma^*$ since $\Lambda(G)$ has no isolated points. Let the number N_H for H be defined by $N_H = \sup_{h \in H_e} \mathrm{ord} h$. And we define the number N'_H as follows: If H_p is non-empty, then $N'_H = +\infty$, otherwise we set $N'_H = N_H$.

For these constants, the next result is fundamental for our present aim.

Theorem 3.1.

$$\min\{\frac{1}{N_H}\lambda_\Gamma, \lambda_H\} \le \lambda_{\tilde{\Gamma}} \le \min\{\lambda_\Gamma, \lambda_H\}, \quad \textit{and} \tag{3.2}$$

$$\min\{\frac{1}{N'_H}\lambda_\Gamma, \lambda^*_H\} \le \lambda^*_{\tilde{\Gamma}} \le \min\{\lambda_\Gamma, \lambda^*_H\}. \tag{3.3}$$

By the relation (2.1), we obtain that $\lambda_\Gamma = L(\Omega)$ and $\lambda_{\tilde{\Gamma}} = L(X)$ and the similar relations for λ^* and L^* hold, hence the above theorem is equivalent to the following

Theorem 3.2.

$$\min\{\frac{1}{N_H}L(\Omega), \lambda_H\} \le L(X) \le \min\{L(\Omega), \lambda_H\}, \quad \textit{and} \tag{3.4}$$

$$\min\{\frac{1}{N'_H}L(\Omega), \lambda^*_H\} \le L^*(X) \le \min\{L(\Omega), \lambda^*_H\}. \tag{3.5}$$

Now we define $\lambda(G), \lambda^*(G), N$ and N' by $\inf_H \lambda_H, \inf_H \lambda^*_H, \sup_H N_H$ and $\sup_H N'_H$, respectively, where H runs over all component subgroups of G. Remark here that the constants $\lambda_H, \lambda^*_H, N_H$ and N'_H are determined only by the conjugacy class of H in G, i.e., by the component $X = \Omega/H$ of $X(G)$. Then, we immediately obtain the following

Theorem 3.3. *For any non-elementary Kleinian group G it follows that*

$$\min\{\frac{1}{N}L(\Omega(G)), \lambda(G)\} \le L(X(G)) \le \min\{L(\Omega(G)), \lambda(G)\}, \quad \textit{and}$$

$$\min\{\frac{1}{N'}L(\Omega(G)), \lambda^*(G)\} \le L^*(X(G)) \le \min\{L(\Omega(G)), \lambda^*(G)\}.$$

Corollary 3.4. *Thus we have $L(\Omega(G)) \ge L^*(X(G))$. In particular, if $X(G)$ is of Lehner type then $\Lambda(G)$ is uniformly perfect.*

Remark. To guarantee that $\lambda(G) > 0$ it is sufficient to assume that $\sup_{z\in\Omega(G)} \iota(z) < \infty$, where $\iota(z)$ denotes the injectivity radius of $\Omega(G)$ at z (oral communication with Prof. Matsuzaki). But, this condition seems to be hard to check.

In the case that G is analytically finite, i.e., $X(G)$ consists of a finite number of Riemann orbifolds of finite type, it is easily verified that $L^*(X(G)) > 0$ thus we have the following collorary.

Corollary 3.5 (Canary [3]). *For an analytically finite non-elementary Kleinian group G, the limit set $\Lambda(G)$ is uniformly perfect.*

We should remark that by Ahlfors' Finiteness Theorem this result produces the finitely generated case.

4. Proof of Theorem 3.1

Under the same situation as the theorem, first we prove the next elementary

Lemma 4.1. *For any element $h \in H$, we have*

$$\inf_{\gamma \in \chi^{-1}(h)} l_\gamma = \min_{\gamma \in \chi^{-1}(h)} l_\gamma = l_h. \tag{4.1}$$

Proof. Let $\gamma \in \chi^{-1}(h)$, then $q \circ \gamma = h \circ q$ by definition. For any $\zeta \in \mathbb{H}$ we put $z = q(\zeta)$. Then, by the definition of metrics, we have

$$d_{\mathbb{H}}(\zeta, \gamma(\zeta)) \geq d_\Omega(q(\zeta), q(\gamma(\zeta))) = d_\Omega(z, h(z)) \geq l_h.$$

Since ζ is arbitrary, it follows that $l_\gamma \geq l_h$. Now we prove the opposite inequality. First, when $l_h = 0$, the covering transformation h covers some loop in X around a singular point or a puncture. Now it is clear that this loop is covered by an elliptic or parabolic element γ in $\widetilde{\Gamma}$, so in any case, we have $l_\gamma = 0$ and $\chi(\gamma) = h$. Next, we assume that $l_h > 0$, then there exists a point $z \in \Omega$ attaining the infimum in (3.1). Choose a point $\zeta \in \mathbb{H}$ with $q(\zeta) = z$, then there exists a geodesic α joining z and $h(z)$ such that $d_\Omega(z, h(z)) = \int_\alpha \rho_\Omega$. Let β be a lift of α via q with initial point ζ, then the terminal point of β can be written as $\gamma(\zeta)$ for some $\gamma \in \widetilde{\Gamma}$. We note here that $\chi(\gamma) = h$ by definition. Therefore, we have

$$d_\Omega(z, h(z)) = \int_\alpha \rho_\Omega = \int_\beta \rho_{\mathbb{H}} \geq d_{\mathbb{H}}(\zeta, \gamma(\zeta)),$$

thus $l_h \geq l_\gamma$. Combining this with the former part of the proof, we have $l_h = l_\gamma$ for some $\gamma \in \chi^{-1}(h)$. Now the proof is completed. □

Now we shall prove the Theorem 3.1. First, we note that if $\gamma \in \widetilde{\Gamma}$ is elliptic, then $h = \chi(\gamma) \in H_e \setminus \{1\}$. Therefore we conclude that

$$\Gamma \cup \chi^{-1}(H \setminus H_e) = \chi^{-1}(1) \cup \chi^{-1}(H \setminus H_e) \subset \widetilde{\Gamma},$$

and this immediately yields the right-hand side inequality in (3.3).

The residual part $\chi^{-1}(H_e \setminus \{1\}) \setminus \widetilde{\Gamma}_e$ which may contribute to the infimum consists of the non-ellipitic element γ such that $h = \chi(\gamma)$ is of finite order n. Therefor $\gamma^n \in \chi^{-1}(1) = \Gamma$, so we have $nl_\gamma = l_{\gamma^n} \geq \lambda_\Gamma$, hence $l_\gamma \geq \frac{1}{n}\lambda_\Gamma \geq \frac{1}{N_H}\lambda_\Gamma$. By this observation, we are convinced the validity of the left-hand side of (3.3). Noting that any parabolic element of $\widetilde{\Gamma}$ is mapped to a parabolic one in H with translation length 0 under the homomorphism χ, we can show the inequality (3.2) in the same way as above.

5. Some consequences

In this section, we shall exhibit several applications of our theorems. We denote by $M_{\Omega(G)}$ ($M^{\circ}_{\Omega(G)}$) the supremum of the moduli of annuli (round annuli, respectively) separating $\Lambda(G)$, where the modulus of an annulus is defined here as the number m when this annulus is conformally equivalent to the round annulus $\{z \in \mathbb{C}, 1 < |z| < e^m\}$ and the round annulus means a bounded annulus with concentric circles as boundary. Further, if $\infty \in \Lambda(G)$, then we can define another constant $C_{\Omega(G)}$ by $\inf_{z\in\Omega(G)} \delta(z)\rho_{\Omega(G)}(z)$, where $\delta(z)$ denotes the Euclidean distance from z to $\Lambda(G)$. For these constants, we know the following estimates.

Theorem 5.1 (cf. [10]). *For a non-elementary Kleinian group G, if we set $L = L(\Omega(G))$ then we obtain that*

$$L \leq \frac{\pi^2}{M_{\Omega(G)}} \leq \min\{Le^L, \frac{1}{2}L^2 \coth^2(\frac{L}{2})\},$$

$$\frac{1}{2}M_{\Omega(G)} - K \leq M^{\circ}_{\Omega(G)} \leq M_{\Omega(G)},$$

where K is an absolute constant $\leq 1.7332\cdots$. Moreover if $\infty \in \Lambda(G)$, it is also valid that

$$\frac{\tanh L/2}{4} \leq C_{\Omega(G)} \leq \frac{\sqrt{3}L}{\sqrt{\pi^2+4L^2}}. \tag{5.1}$$

In particular, $L(\Omega(G)) > 0$ if and only if $M_{\Omega(G)} < \infty$.

Furthermore, Pommerenke has given a remarkable characterization of the uniform perfectness in the words of capacity density.

Theorem 5.2 (Pommerenke [8]). *$\Lambda(G)$ is uniformly perfect if and only if there exists a constant $c \in (0, 1]$ such that $\mathrm{Cap}(\Lambda(G) \cap B(a,r)) \geq cr$ for any $a \in \Lambda(G)$ and $0 < r \leq \mathrm{diam}(\Lambda(G))$, where Cap denotes the logarithmic capacity and $B(a,r)$ the closed disk centered at a with radius r.*

We note that $\mathrm{Cap}(B(a,r)) = r$ and $2^{-7}e^{-M^{\circ}_{\Omega(G)}}$ can be taken as the constant c (see [10]). In particular, by virtue of Wiener's criterion, we have the following

Corollary 5.3. *If $L(\Omega(G)) > 0$ the limit set $\Lambda(G)$ is regular in the sense of Dirichlet.*

For general Kleinian groups, at least we can state the following

Corollary 5.4. *Let G be a non-elementary Kleinian group. Any loxodromic or parabolic fixed point of G is a regular point of $\Lambda(G)$ in the sense of Dirichlet.*

In fact, if z_0 is a fixed point of a loxodromic or parabolic element γ of G, then z_0 is contained by the limit set of some finitely generated non-elementary subgroup G_0 of G containing γ. Since $\Lambda(G_0)$ is uniformly perfect, by Theorem 5.2, we see that $\overline{\lim}_{r\to 0}\mathrm{Cap}(\Lambda(G)\cap B(z_0,r))/r \geq \overline{\lim}_{r\to 0}\mathrm{Cap}(\Lambda(G_0)\cap B(z_0,r))/r > 0$, which implies the regularity of $\Lambda(G)$ at z_0 (see, for example, Tsuji [11] p.104). Further, we should note that the set of loxodromic fixed points of any non-elementary Kleinian group is dense in the limit set.

Another application of unifom perfectness is concerned with the Hausdorff dimension. The next result is essentially due to Järvi-Vuorinen [4].

Theorem 5.5 (cf. [10]). *The Hausdorff dimension* H-dim$(\Lambda(G))$ *of* $\Lambda(G)$ *can be estimated from below as follows.*

$$\text{H-dim}(\Lambda(G)) \geq \frac{\log 2}{\log(2e^{M^\circ_{\Omega(G)}}+1)} \left(\geq \frac{\log 2}{M^\circ_{\Omega(G)}+\log 3}\right).$$

As an immediate consequence of this, we can see that any non-elementary Kleinian group has the limit set of positive Hausdorff dimemsion.

6. Estimate of translation length by the trace

As an application of (5.1), we present here an estimate of the translation length of a loxodromic element of Kleinian group in the region of discontinuity by its trace. Let G be a non-elementary Kleinian group with $L(\Omega(G)) > 0$ and H a component subgroup of G corresponding to a component Ω of the region of discontinuity $\Omega(G)$. We set $C = \frac{1}{4}\tanh(L(\Omega(G))/2) > 0$. And let h be a loxodromic element of H, i.e., $\tau = \mathrm{tr}^2(h) \in \mathbb{C}\setminus[0,4]$. By the Möbius-invariance of l_h and $\mathrm{tr}^2(h)$, we may assume that h has the form $h(z) = \lambda z$, where $\tau = (\sqrt{\lambda}+\sqrt{\lambda}^{-1})^2 = \lambda+\lambda^{-1}+2$. In this time, we note that $0,\infty \in \Lambda(G)$. We denote by Log$z$ the principal branch of $\log z$, i.e., Log$z$ is the branch of log determined by $-\pi <$ ImLog$z \leq \pi$. Let $\zeta = \mathrm{Log}\lambda = \pm\mathrm{Log}\frac{\tau-2+\sqrt{(\tau-2)^2-4}}{2}$. If we denote by $\delta(z)$ the distance from z to $\Lambda(G)$, by (5.1), we have $\delta(z)\rho_\Omega(z) \geq C$ for $z \in \Omega$ and $\delta(z) \leq |z|$ because $0 \in \Lambda(G)$. Let α be a geodesic arc in Ω from z_0 to $h(z_0) = \lambda z_0$ with minimal length and β a lift of α via the exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$. We denote by w_0, w_1 the initial and terminal point of β. By construction, the difference $w_1 - w_0$ is a logarithm of λ, so we note here $|w_1 - w_0| \geq |\mathrm{Log}\lambda|$. Then, we compute

$$\begin{aligned} d_\Omega(z_0, h(z_0)) &= \int_\alpha \rho_\Omega(z)|dz| \geq C\int_\alpha \frac{|dz|}{\delta(z)} \geq C\int_\alpha \frac{|dz|}{|z|} \\ &= C\int_\beta |dw| \geq C|w_1 - w_0| \geq C|\mathrm{Log}\lambda|. \end{aligned}$$

Since z_0 was arbitrary, we have $l_h \geq C|\mathrm{Log}\lambda|$. Now we have proved the next

Proposition 6.1. *Let H be a component subgroup of a non-elementary Kleinian group G and suppose that $C = \frac{1}{4}\tanh(L(\Omega(G))/2) > 0$. Then, for any loxodromic element $h \in H$, the translation length l_h can be estimated as*

$$l_h \geq C|\mathrm{Log}\lambda|,$$

where λ is the multiplier of h, i.e., $\mathrm{tr}^2(h) = \lambda + \lambda^{-1} + 2$.

Remark. In general, we cannot estimate the translation length from above by the trace. This can be understood by the existence of accidental parabolic transformations.

7. Example

In this concluding section, we present a simple example of (infinitely generated) Kleinian groups with non-uniformly perfect limit sets. Looking at the Theorem 3.3, we can guess that if quotient orbifold has arbitrarily short geodesics which are lifted to non-closed curves in the region of discontinuity. In fact, such an example can be given by infinitely generated Schottky groups as Pommerenke indicated in [8].

Let $a_j, b_j \in \mathbb{C}$ be sequences tending to ∞, and $r_j > 0$ and $\alpha_j \in \mathbb{C}$ with $|\alpha_j| = 1$ be given so that all closed disks $A_j = B(a_j, r_j), B_j = B(b_j, r_j)$ are disjoint $(j = 1, 2, \dots)$. We set $g_j(z) = b_j - \frac{\alpha_j r_j^2}{z - a_j}$, then A_j and B_j are the isometric circles for g_j and g_j^{-1}, respectively, thus $G = \langle g_1, g_2, \dots \rangle$ is an infinitely generated Schottky group with a fundamental domain $\mathbb{C} \setminus \cup_j (A_j \cup B_j) \subset \Omega(G)$.

Now we set $\tilde{r}_j = \mathrm{dist}(a_j, (\cup_{k \neq j} A_k) \cup (\cup_k B_k)) > r_j$. Then, we can directly see that $M^{\circ}_{\Omega(G)} \geq \sup_j \tilde{r}_j / r_j$, hence $\Lambda(G)$ is not uniformly perfect if $\sup_j \tilde{r}_j / r_j = \infty$.

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DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, 606-01 SAKYO-KU, KYOTO CITY, JAPAN

E-mail address: sugawa@kusm.kyoto-u.ac.jp